

Mathematics 100A Quiz 4

Due: Friday 20 November 2020

Instructions: This is a closed note, closed book quiz. Please write clearly and fully explain your solutions. You have 30 minutes from when you begin to read the questions to complete it.

For both questions, we use the following notation: For a vector $a \in \mathbf{R}^2$, let t_a denote translation by a . Similarly, for $\theta \in \mathbf{R}$, let ρ_θ denote the isometry of $\mathbf{R}^2$ that is rotation by the angle θ about the origin.

1. Suppose $f = t_a \rho_{2\pi/6}$, an isometry of $\mathbf{R}^2$. Prove that $f^6 = f \circ f \circ f \circ f \circ f \circ f$ is a translation.

Hint: Use the group homomorphism $M_2 \rightarrow O_2$ that we have discussed.

Proof. Let $\varphi : M_2 \rightarrow O_2$ denote the map defined as $\varphi(t_a g) = g$ if $g \in O_2$. As we proved, this map φ is a group homomorphism with kernel the translations. For the f given in the problem, $\varphi(f^6) = \varphi(f)^6 = \rho_{2\pi/6}^6 = 1$. Thus f^6 is in the kernel of φ , so is a translation. $\square$

2. Let $P = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be a point in $\mathbf{R}^2$. Let f denote the isometry of $\mathbf{R}^2$ that is rotation by the angle $\pi/2$ about the point P . Write f in the form $t_b \rho_\theta$ for explicitly defined b and θ .

Proof. We have $f = t_P \rho_{\pi/2} t_{-P}$. As $\rho t_v = t_{\rho(v)} \rho$, we obtain $f = t_{P - \rho_{\pi/2}(P)} \rho_{\pi/2}$. Thus $\theta = \pi/2$ and

$$b = P - \rho_{\pi/2}(P) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

$\square$